# Multiagent decision-making and control

## Games in extensive form

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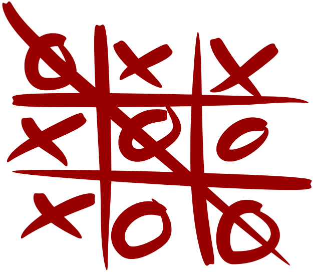
# Course topics

1. Static games
2. Zero-sum games
3. Potential games
4. Extensive form games
5. Dynamic games, dynamic programming principle
6. Dynamic games, dynamic programming for games
7. Dynamic games, linear quadratic games, Markov games
8. Convex games, Nash equilibria characterization
9. Convex games, Nash equilibria computation
10. Auctions
11. Bayesian games
12. Learning in games
13. Final project presentations

# Example: Tic-tac-toe

- **Player 1** draws $\times$ on the board.
- **Player 2** draws $\circ$ on the board.
- The game ends when either Player 1 or Player 2 has put **three signs in a row**, or when **the board is full**.



- What are the differences between this game and the games we have seen previously?

# “Advanced” games

Features:

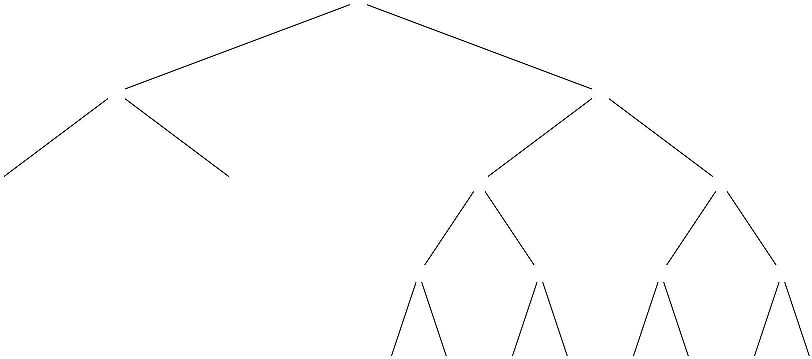
- different order of play
- multiple stages
- variable number of stages
- partial information (dependent on actions)
- memory-constrained players

The **matrix form** is not the most effective representation.

# Extensive form

A finite **tree** structure where

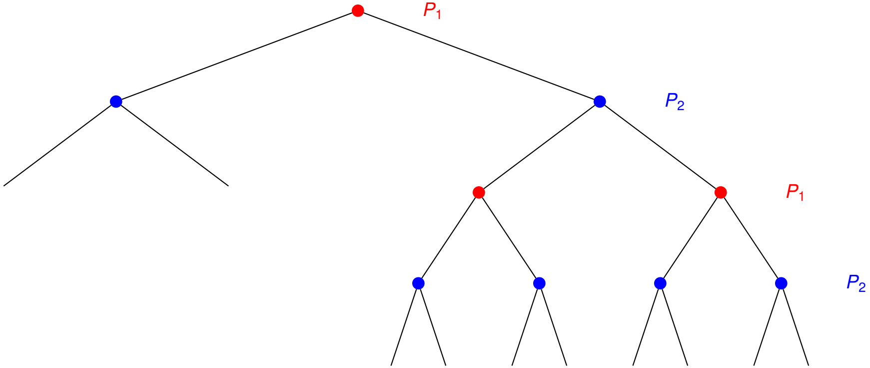
- the game evolves from the **root** to the **leaves**
  - Let us consider two-player games



# Extensive form

A finite **tree** structure where

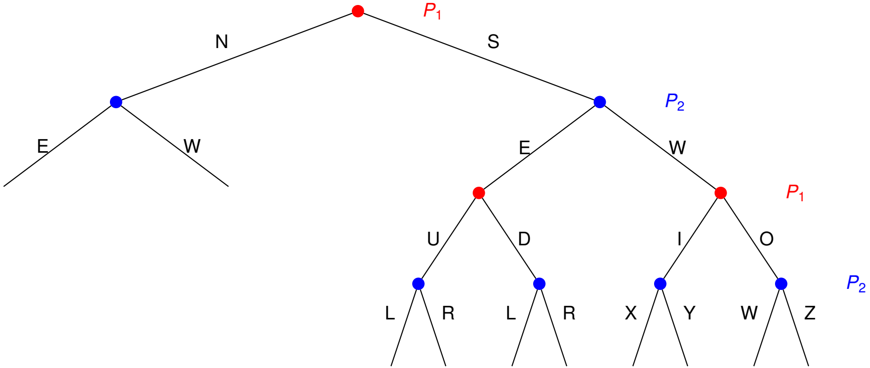
- each **level** of the tree corresponds to a player's turn
  - A **stage** is made of two turns, i.e. two **levels** of the tree



# Extensive form

A finite **tree** structure where

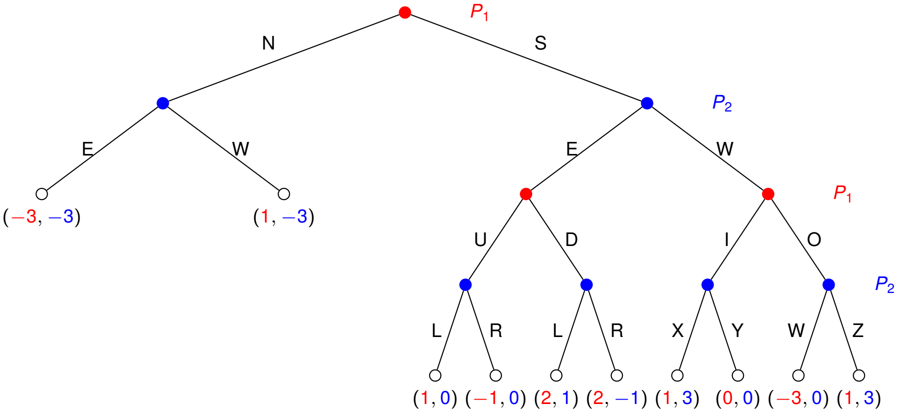
- **links** correspond to actions



# Extensive form

A finite **tree** structure where

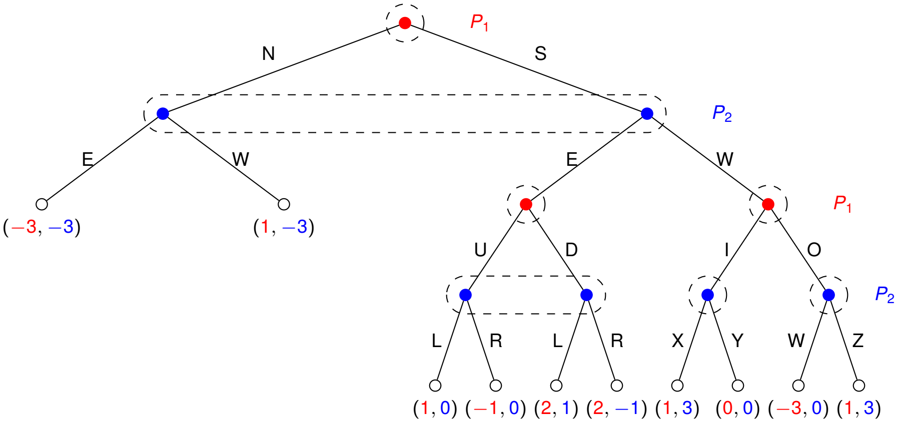
- each leaf is associated to an **outcome**



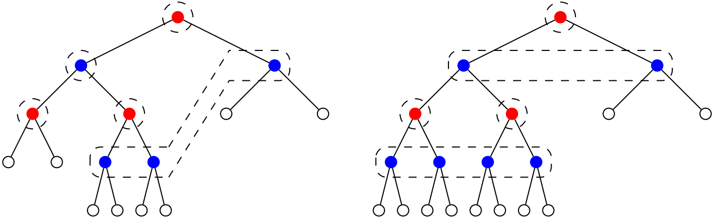
# Extensive form

A finite **tree** structure where

- nodes of each player are divided into **information sets**
  - each node in the same information set has the same branches



Some exotic information sets...



# Actions and strategies

Action $\neq$ Strategy!

Let $\mathcal{I}_1, \ldots, \mathcal{I}_r$ be the **information sets** of Player 1.

Let $\mathcal{U}_i$ be the set of **actions** available to player 1 in the information set $\mathcal{I}_i$.

A **pure strategy** $\gamma = (\gamma_1, \ldots, \gamma_r)$ for player 1 is **a map** that assigns an action to each information set.

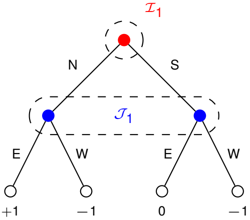
$$\gamma_i : \mathcal{I}_i \to \mathcal{U}_i$$

Similarly, for player 2
Let $\mathcal{J}_1, \ldots, \mathcal{J}_s$ be the **information sets** of player 2. Let $\mathcal{V}_i$ be the set of **actions** available to Player 2 in the information set $\mathcal{J}_i$.
What is a pure strategy for player 2?

$$\sigma = (\sigma_1, \ldots, \sigma_s)$$

$$\sigma_i : \mathcal{J}_i \to \mathcal{V}_i, \quad i = 1, 2, \ldots, s$$

# Actions and strategies - example (zero-sum game)




IS : $\{\mathcal{I}_1\}$ IS : $\{\mathcal{J}_1\}$

$\mathcal{U}_1 = \{N, S\}$ $\mathcal{V}_1 = \{E, W\}$

Example pure strategy

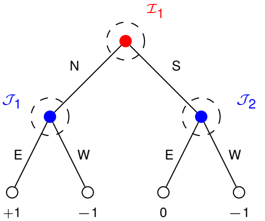
$$\gamma(\mathcal{I}_1) = N$$
$$\sigma(\mathcal{J}_1) = E$$

Example of a single-stage game.

- Player 1 plays "first"
- Player 2 plays "second"
- Player 2 does not know the action of Player 1

**Simultaneous play**

# Actions and strategies




$IS : \{\mathcal{I}_1\}$ $\qquad$ $IS : \{\mathcal{J}_1, \mathcal{J}_2\}$

$\mathcal{U}_1 = \{N, S\}$ $\qquad$ $\mathcal{V}_1 = \mathcal{V}_2 = \{E, W\}$

Example pure strategy

$$\gamma(\mathcal{I}_1) = N$$

$$\sigma(\mathcal{J}_1) = E, \quad \sigma(\mathcal{J}_2) = W$$

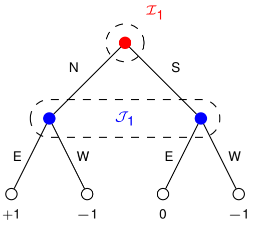
$$\sigma = (\sigma_1, \sigma_2)$$

Example of a single-stage game.

- Player 1 plays first
- Player 2 plays second
- Player 2 knows the action of Player 1
- $\mathcal{V}_1$ and $\mathcal{V}_2$ (i.e., possible actions in $\mathcal{J}_1$ and in $\mathcal{J}_2$) could be different!

**Sequential play**

# From extensive form to matrix form




$P1 : \{\mathcal{I}_1\}$ $\quad$ $P2 : \{\mathcal{J}_1\}$

$\mathcal{U}_1 = \{N, S\}$ $\quad$ $\mathcal{V}_1 = \{E, W\}$

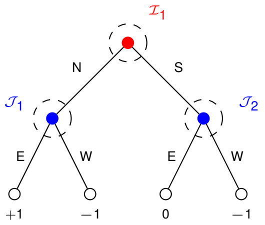
**(simultaneous play)**

**Remember:** each row/column corresponds to a **strategy**

| | $\sigma(\mathcal{J}_1) = E$ | $\sigma(\mathcal{J}_1) = W$ |
|---|---|---|
| $\gamma(\mathcal{I}_1) = N$ | $+1$ | $-1$ |
| $\gamma(\mathcal{I}_1) = S$ | 0 | $-1$ |

Nash equilibria

# From extensive form to matrix form




$P1 : \{\mathcal{I}_1\}$ $P2 : \{\mathcal{J}_1, \mathcal{J}_2\}$

$\mathcal{U}_1 = \{N, S\}$ $\mathcal{V}_1 = \mathcal{V}_2 = \{E, W\}$

**(sequential play)**

**Remember:** each row/column corresponds to a **strategy**

| | $\sigma(\mathcal{J}_1) = E$ <br> $\sigma(\mathcal{J}_2) = E$ | $\sigma(\mathcal{J}_1) = E$ <br> $\sigma(\mathcal{J}_2) = W$ | $\sigma(\mathcal{J}_1) = W$ <br> $\sigma(\mathcal{J}_2) = E$ | $\sigma(\mathcal{J}_1) = W$ <br> $\sigma(\mathcal{J}_2) = W$ |
|---|---|---|---|---|
| $\gamma(\mathcal{I}_1) = N$ | $+1$ | $+1$ | $-1$ | $-1$ |
| $\gamma(\mathcal{I}_1) = S$ | $0$ | $-1$ | $0$ | $-1$ |

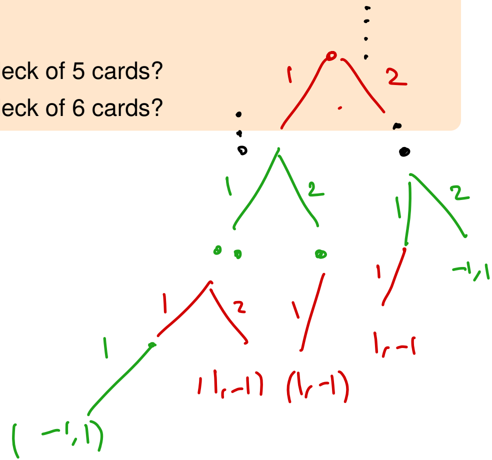
# Game play: Card playing

Consider the following card game with a deck of 4 cards.

- Players take turn. At each time a player can take 1 or 2 cards.
- The player who takes the last card will win

**Activity - get in groups of two and do the following:**

- Play the game
- Draw the game tree
- How would you play the game for deck of 5 cards?
- How would you play the game for deck of 6 cards?



# From extensive form to matrix form

**Games in extensive form can be reformulated in matrix form!**

Definitions/results extend to this larger class of games, as long as we use the new definition of *strategy*.

Example: **Saddle point equilibria (pure equilibrium strategy)**

*Definition:* A pair of strategies $\gamma^*$ and $\sigma^*$ is a saddle point equilibrium if for any other policies $\gamma$ and $\sigma$, the outcome $J(\gamma, \sigma)$ of the game satisfies

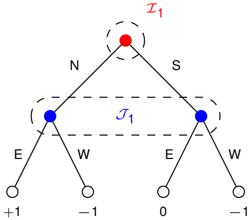
$$J(\gamma^*, \sigma) \quad \leq \quad \underbrace{J(\gamma^*, \sigma^*)}_{\text{saddle-point value of the game}} \quad \leq \quad J(\gamma, \sigma^*)\,.$$

Let us check the previous examples.

# From extensive form to matrix form

0. sum ,

P1 is minimizer

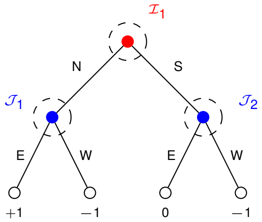



- The strategy $\sigma(\mathcal{J}_1) = W$ is **strictly dominated** by $\sigma(\mathcal{J}_1) = E$
- The strategy $\gamma(\mathcal{I}_1) = N$ is **strictly dominated** by $\gamma(\mathcal{I}_1) = S$
- **One pure NE**

| | $\sigma(\mathcal{J}_1) = E$ | $\sigma(\mathcal{J}_1) = W$ |
|---|---|---|
| $\gamma(\mathcal{I}_1) = N$ | $+1$ | $-1$ |
| $\gamma(\mathcal{I}_1) = S$ | $0$ | $-1$ |

Nash

# From extensive form to matrix form




What are the Nash equilibrium strategies?

| | $\sigma(\mathcal{J}_1) = E$ <br> $\sigma(\mathcal{J}_2) = E$ | $\sigma(\mathcal{J}_1) = E$ <br> $\sigma(\mathcal{J}_2) = W$ | $\sigma(\mathcal{J}_1) = W$ <br> $\sigma(\mathcal{J}_2) = E$ | $\sigma(\mathcal{J}_1) = W$ <br> $\sigma(\mathcal{J}_2) = W$ |
|---|---|---|---|---|
| $\gamma(\mathcal{I}_1) = N$ | $+1$ | $+1$ | $-1$ | $-1$ |
| $\gamma(\mathcal{I}_1) = S$ | $0$ | $-1$ | $0$ | $-1$ |

Nash equilibrium

# From extensive form to matrix form

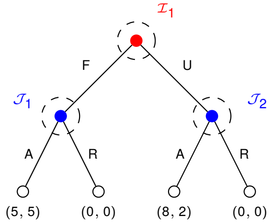
### Computing Nash equilibria

**Idea:** Find equilibrium strategies in the matrix form of the extensive game.

**Problem 1:** Scalability: the size of the matrix is exponential in number of information sets
**Problem 2:** Some equilibria do not capture the timing of the game play.

# An ultimatum game

Player 1 makes a fair or unfair offer, player 2 accepts or rejects
A: accept, R: reject, F: fair, U: unfair




- What are the pure Nash equilibria?
- How would you play this game?

players are maximizers here

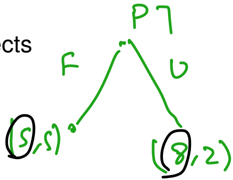
Nash equilibria

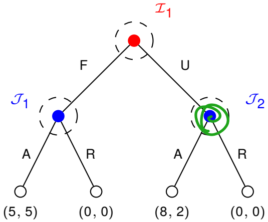
| | $\sigma(\mathcal{J}_1) = A$ $\sigma(\mathcal{J}_2) = A$ | $\sigma(\mathcal{J}_1) = A$ $\sigma(\mathcal{J}_2) = R$ | $\sigma(\mathcal{J}_1) = R$ $\sigma(\mathcal{J}_2) = A$ | $\sigma(\mathcal{J}_1) = R$ $\sigma(\mathcal{J}_2) = R$ |
|---|---|---|---|---|
| $\gamma(\mathcal{I}_1) = F$ | $(5, 5)$ | $(5, 5)$ | $(0, 0)$ | $(0, 0)$ |
| $\gamma(\mathcal{I}_1) = U$ | $(8, 2)$ | $(0, 0)$ | $(8, 2)$ | $(0, 0)$ |

Ultimatum game was introduced by Nobel laureate John Harsanyi

# An ultimatum game

Player 1 makes a fair or unfair offer, player 2 accepts or rejects
A: accept, R: reject, F: fair, U: unfair




### Subgame perfect NE

A strategy is a **subgame perfect equilibrium** if it represents a NE of every subgame of the original game.

players are maximizers here

Which one is a subgame perfect equilibrium?

| | $\sigma(\mathcal{J}_1) = A$ $\sigma(\mathcal{J}_2) = A$ | $\sigma(\mathcal{J}_1) = A$ $\sigma(\mathcal{J}_2) = R$ | $\sigma(\mathcal{J}_1) = R$ $\sigma(\mathcal{J}_2) = A$ | $\sigma(\mathcal{J}_1) = R$ $\sigma(\mathcal{J}_2) = R$ |
|---|---|---|---|---|
| $\gamma(\mathcal{I}_1) = F$ | $(5, 5)$ | $(5, 5)$ | $(0, 0)$ | $(0, 0)$ |
| $\gamma(\mathcal{I}_1) = U$ | $(8, 2)$ | $(0, 0)$ | $(8, 2)$ | $(0, 0)$ |

Ultimatum game was introduced by Nobel laureate John Harsanyi

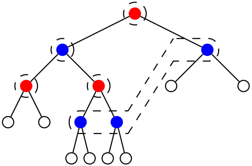
# Notion of subgame

Subgame is a part of the game that is itself a well-defined game

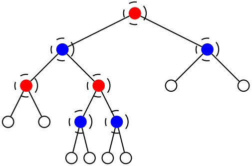
- the initial node is in a singleton information set
- if a node is contained in the subgame then so are all of its successors
- if a node in an information set is in the subgame then all nodes in that information set are also in the subgame

# Subgame perfect equilibria

- The notion of **subgame** is not always well defined.

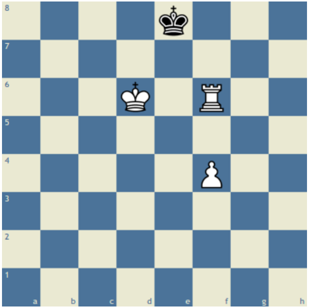


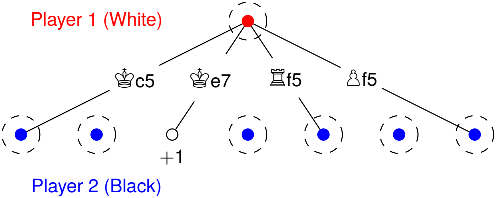
- One special case: **games with perfect information**.



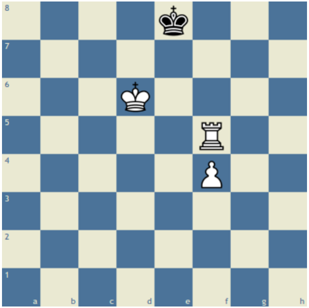
# Chess - a game with perfect information

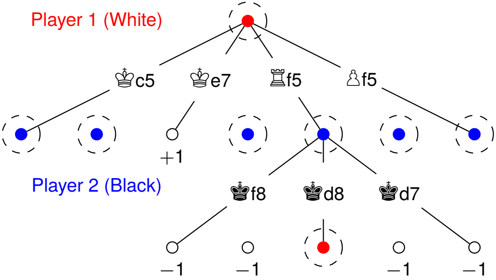
In case of games with **perfect information**, we can search for a pure NE more efficiently.



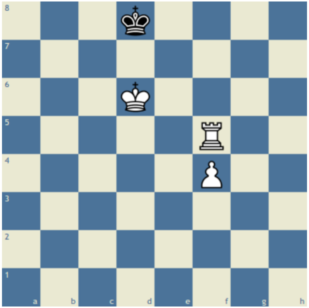


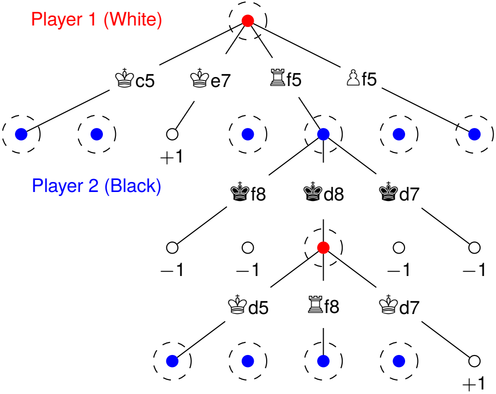
# Chess


8
7
6
5
4
3
2
1
a b c d e f g h

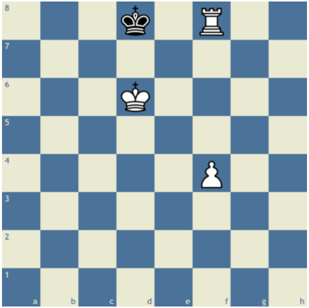

Player 1 (White)
♔c5
♔e7
♖f5
♙f5
+1
Player 2 (Black)
♚f8
♚d8
♚d7
−1
−1
−1
−1

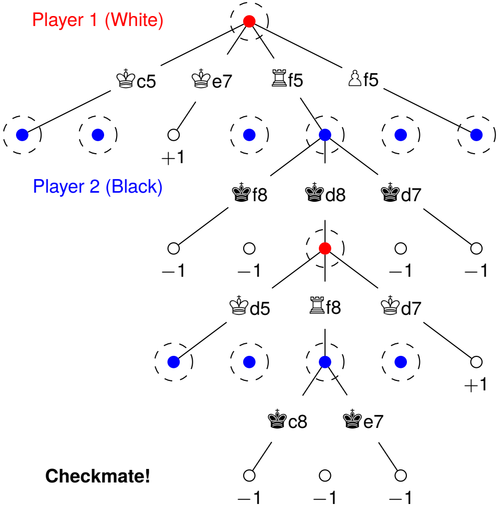
# Chess




Player 1 (White)
♔c5
♔e7
♖f5
♙f5
+1
Player 2 (Black)
♚f8
♚d8
♚d7
−1
−1
−1
−1
♔d5
♖f8
♔d7
+1

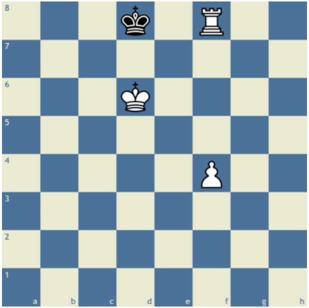
# Chess

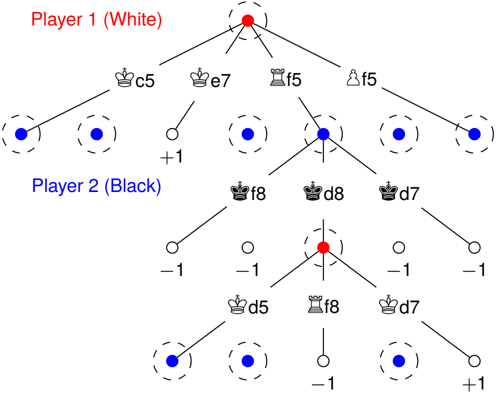




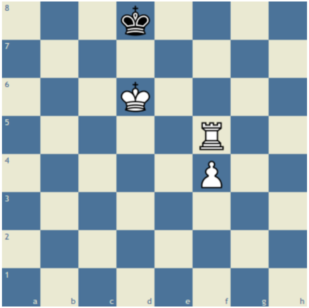
Player 1 (White)
♔c5
♔e7
♖f5
♙f5
+1
Player 2 (Black)
♚f8
♚d8
♚d7
−1
−1
−1
−1
♔d5
♖f8
♔d7
+1
♚c8
♚e7
Checkmate!
−1
−1
−1

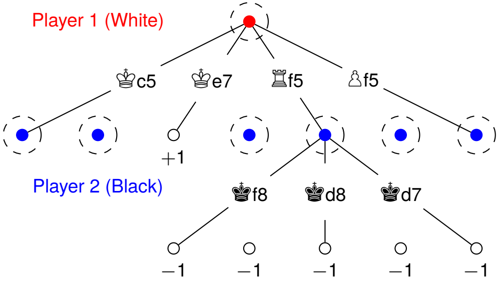
# Backward induction



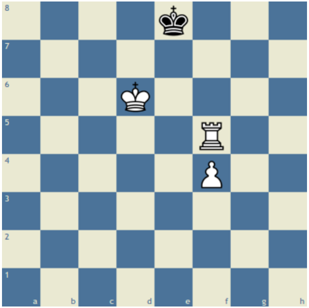


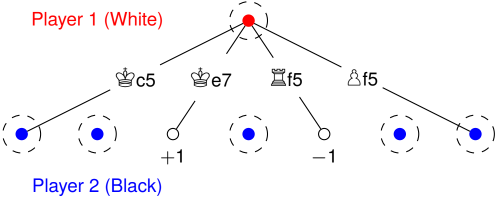
# Backward induction



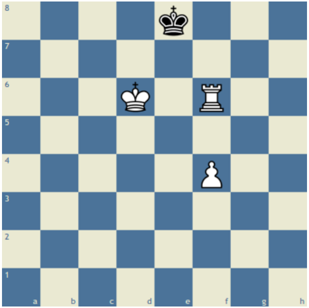


# Backward induction





# Backward induction



Player 1 (White) ○ $-1$

The game was decided since the beginning!

"White checkmate in two"

# Zermelo's theorem (1913)

Different versions of the statement - see here. Chess is a two-person game without chance, where players have strictly opposing interests and where only a finite number of positions are possible.

Therefore one of the following must be true:

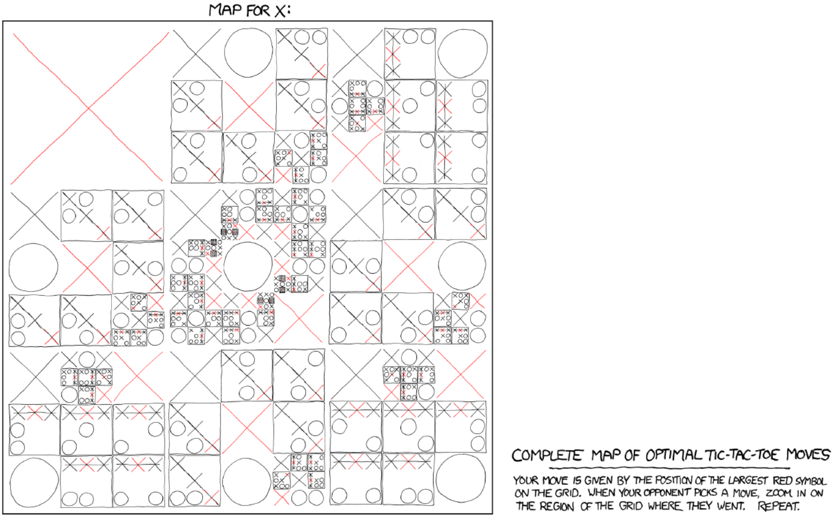
- the first-player can force a win
- the second-player can force a win
- both players can at least force a draw

  *"Should it be answered exactly, Chess would of course lose the character of a game at all. "*

  *It is thus the sheer complexity of the game of Chess alone that allows it to retain its mystery.*

# Backward induction

In games with perfect information, we can solve for subgame perfect Nash equilibrium using backward induction.

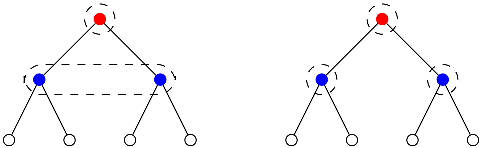



xkcd.com

# Backward induction

## Features

- More **efficient** than exploring the matrix form
- Returns a strategy that is a **subgame perfect equilibrium**



Can we always apply backward induction?

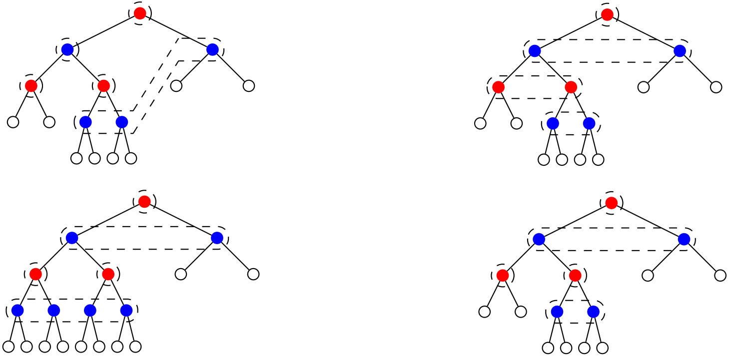
We can for games with **perfect information.**

What other games allow the use of backward induction?

# Feedback games

A multi-stage game in extensive form is a **feedback game** if

1. no information set spans over multiple stages
2. each “Player 1” node is the root of a separated sub-game.



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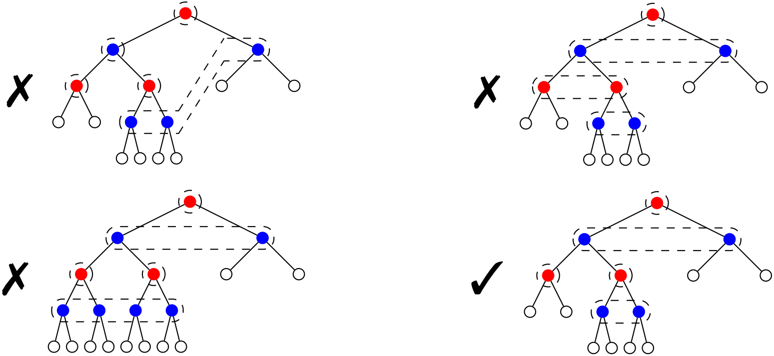
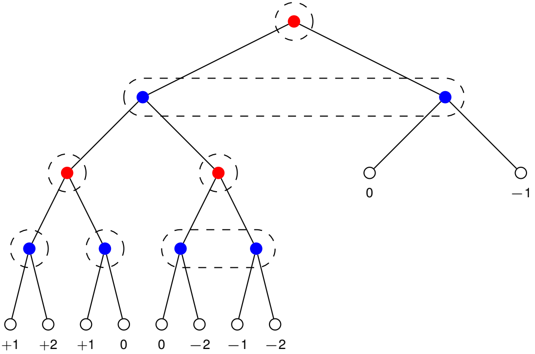


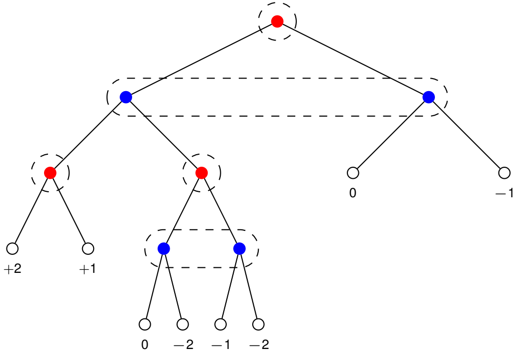
# Backward induction in feedback games




Starting from the leaves, identify **subgames** for which we can determine the **pure** NE strategy...

... either because players have **full information**...

# Backward induction

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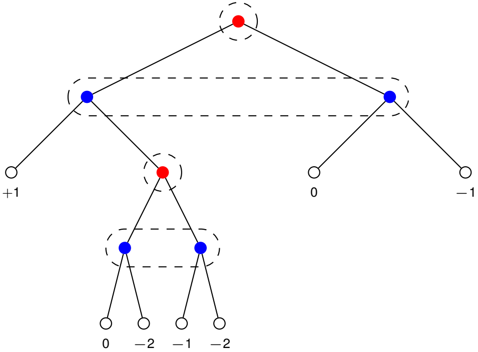
... either because players have **full information**...

# Backward induction

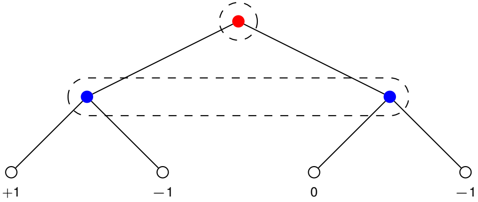



Starting from the leaves, identify **subgames** for which we can determine the **pure** NE strategy...

... either because players have **full information**...

... or because they play a **simultaneous game with pure NE**.

# Backward induction



# Backward induction


- Solve the game **from the leaves towards the root**
- Move up **stage-by-stage** (not level-by-level)
- Record the **pure** equilibrium strategy **for each information set**
- **If the algorithm converges to the root**, then we have a pure NE.
- **No guarantees of convergence**, even when a pure NE exists.

# Formalizing backward induction for feedback games

We can use backward induction to compute subgame perfect equilibria, if they exist. How? And Why?

Read: Sections 7.7-7.9 of Hespanha book.
Based on the reading, complete the following tasks for zero-sum feedback games.

1. Definition: pure strategy subgame perfect equilibria for feedback games. These are known also as pure feedback saddle-point equilibria.
2. Backward induction: define the approach to compute a pure strategy subgame perfect equilibria for feedback games
3. Why the strategy found from backward induction above is a Nash equilibrium of the multi-stage feedback game?
4. How can this approach be generalized to non zero-sum feedback games?

# Backward induction may not find all pure NE

Read: Example from Figure 7.4 of Hespanha book.

# Summary

- Extensive form games and game tree
- Information sets, strategies
- Pure strategy equilibria for extensive form games
- Converting extensive form games to matrix games
- Pure strategy subgame perfect equilibria
- Examples: tic-tac-toe, chess, card playing
- Backward induction

games with perfect informat

chess, .